

# On modules over valuations.

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## Abstract

To any smooth manifold  $X$  an algebra of smooth valuations  $V^\infty(X)$  was associated in [1]-[3],[5]. In this note we initiate a study of  $V^\infty(X)$ -modules. More specifically we study finitely generated projective modules in analogy to the study of vector bundles on a manifold. In particular it is shown that for a compact manifold  $X$  there exists a canonical isomorphism between the  $K$ -ring constructed out of finitely generated projective  $V^\infty(X)$ -modules and the classical topological  $K^0$ -ring constructed out of vector bundles over  $X$ .

## 1 Introduction.

Let  $X$  be a smooth manifold of dimension  $n$ .<sup>1</sup> In [1]-[3],[5] the notion of a smooth valuation on  $X$  was introduced. Roughly put, a smooth valuation is a  $\mathbb{C}$ -valued finitely additive measure on compact submanifolds of  $X$  with corners, which satisfies in addition some extra conditions. We omit here the precise description of the conditions due to their technical nature. Let us notice that basic examples of smooth valuations include any smooth measure on  $X$  and the Euler characteristic. There are many other natural examples of valuations coming from convexity, integral, and differential geometry. We refer to recent lecture notes [4], [8], [7] for an overview of the subject, examples, and applications.

The space  $V^\infty(X)$  of all smooth valuations is a Fréchet space. It has a canonical product making  $V^\infty(X)$  a commutative associative algebra over  $\mathbb{C}$  with a unit element (which is the Euler characteristic).

In this note we initiate a study of modules over  $V^\infty(X)$ . Our starting point is the analogy to the following well known fact due to Serre and Swan [11], [12]: if  $X$  is compact, then the category of smooth vector bundles of finite rank over  $X$  is equivalent to the category of finitely generated projective modules over the algebra  $C^\infty(X)$  of smooth functions (the functor in one direction is given by taking global smooth sections of a vector bundle).

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<sup>1</sup>All manifolds are assumed to be countable at infinity, i.e. presentable as a union of countably many compact subsets. In particular they are paracompact.

In order to state our main results we need to remind a few general facts about valuations on manifolds. We have a canonical homomorphism of algebras

$$V^\infty(X) \rightarrow C^\infty(X) \quad (1)$$

given by the evaluation on points, i.e.  $\phi \mapsto [x \mapsto \phi(\{x\})]$ . This is an epimorphism. The kernel, denoted by  $W_1$ , is a nilpotent ideal of  $V^\infty(X)$ :

$$(W_1)^{n+1} = 0.$$

Next, smooth valuations form a sheaf of algebras which is denoted by  $\mathcal{V}_X^\infty$ : for an open subset  $U \subset X$ ,

$$\mathcal{V}_X^\infty(U) = V^\infty(U),$$

where the restriction maps are obvious. We denote by  $\mathcal{O}_X$  the sheaf of  $C^\infty$ -smooth functions on  $X$ . Then the map (1) gives rise to the epimorphism of sheaves

$$\mathcal{V}_X^\infty \twoheadrightarrow \mathcal{O}_X. \quad (2)$$

Recall now the notion of a projective module. Let  $A$  be a commutative associative algebra with a unit. An  $A$ -module  $M$  is called *projective* if  $M$  is a direct summand of a free  $A$ -module, i.e. there exists an  $A$ -module  $N$  such that  $M \oplus N$  is a free  $A$ -module (not necessarily of finite rank). It is easy to see that if  $M$  is in addition finitely generated then  $M$  is a direct summand of a free  $A$ -module of finite rank.

Let  $\mathcal{A}$  be a sheaf of algebras on a topological space  $X$ . A sheaf  $\mathcal{M}$  of  $\mathcal{A}$ -modules is called a *locally projective*  $\mathcal{A}$ -module if any point  $x \in X$  has an open neighborhood  $U$  such that  $\mathcal{M}(U)$  is a projective  $\mathcal{A}(U)$ -module.

Let us denote by  $Proj_f \mathcal{V}_X^\infty - mod$  the full subcategory of  $\mathcal{V}_X^\infty$ -modules consisting of locally projective  $\mathcal{V}_X^\infty$ -modules of finite rank. Let us denote by  $Proj_f V^\infty(X) - mod$  the full subcategory of the category of  $V^\infty(X)$ -modules consisting of projective  $V^\infty(X)$ -modules of finite rank. In Section 2 we prove the following result.

**1.1 Theorem.** *Let  $X$  be a smooth manifold.*

- (1) *Any locally projective  $\mathcal{V}_X^\infty$ -module of finite rank is locally free.*
- (2) *Assume in addition that  $X$  is compact. Let  $\mathcal{E}$  be a locally free  $\mathcal{V}_X^\infty$ -module of finite rank. Then there exists another locally free  $\mathcal{V}_X^\infty$ -module  $\mathcal{H}$  of finite rank such that  $\mathcal{E} \oplus \mathcal{H}$  is isomorphic to  $(\mathcal{V}_X^\infty)^N$  for some natural number  $N$ .*
- (3) *Assume again that  $X$  is compact. Then the functor of global sections*

$$\Gamma: Proj_f \mathcal{V}_X^\infty - mod \rightarrow Proj_f V^\infty(X) - mod$$

*is an equivalence of categories.*

Notice that all the statements of the theorem are completely analogous to the classical situation of vector bundles (whose spaces of sections are projective finitely generated  $C^\infty(X)$ -modules). For example a version of (2) for vector bundles says that any vector bundle is a direct summand of a free bundle. A classical version of (3) is the above mentioned theorem

of Serre-Swan. The method of proof of Theorem 1.1 is a minor modification of the proof for the analogous statement for vector bundles.

To formulate our next main result observe that to any  $\mathcal{V}_X^\infty$ -module we can associate an  $\mathcal{O}_X$ -module via

$$\mathcal{M} \mapsto \mathcal{M} \otimes_{\mathcal{V}_X^\infty} \mathcal{O}_X, \quad (3)$$

where  $\mathcal{O}_X$  is considered as  $\mathcal{V}_X^\infty$ -module via the epimorphism (2). Clearly under this correspondence locally free  $\mathcal{V}_X^\infty$ -modules of finite rank are mapped to locally free  $\mathcal{O}_X$ -modules of equal rank, i.e. to vector bundles.

**1.2 Theorem.** *Assume that  $X$  is a compact manifold. Let  $N$  be a natural number. The map (3) induces a bijection between the isomorphism classes of locally free  $\mathcal{V}_X^\infty$ -modules of rank  $N$  and isomorphism classes of vector bundles of rank  $N$ .*

Theorem 1.2 is proved in Section 3. The proof is an application of general results of Grothendieck [9] on non-abelian cohomology of topological spaces and the existence of a finite decreasing filtration on  $\mathcal{V}_X^\infty$  such that the associated graded sheaf is a sheaf of  $\mathcal{O}_X$ -modules.

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## 2 Locally free sheaves over valuations.

A sheaf  $\mathcal{E}$  of  $\mathcal{V}_X^\infty$ -modules is called locally projective of finite rank if every point  $x \in X$  has a neighborhood  $U$  such that there exists a sheaf of  $\mathcal{V}_U^\infty$ -modules  $\mathcal{F}$  with the property that  $\mathcal{E}|_U \oplus \mathcal{F}$  is isomorphic to  $(\mathcal{V}_X^\infty)^N$  for some natural number  $N$ .

The technique used in the proofs of most of the results of this section is rather standard and is a simple modification of that from [6].

**2.1 Proposition.** *On a manifold  $X$  any locally projective  $\mathcal{V}_X^\infty$ -module of finite rank is locally free.*

**Proof.** Fix a point  $x_0 \in X$ . Let us denote for brevity  $\mathcal{V}_{x_0} := \mathcal{V}_{X,x_0}^\infty$  (resp.  $\mathcal{O}_{X,x_0}$ ) the stalk at  $x_0$  of the sheaf  $\mathcal{V}_X^\infty$  (resp.  $\mathcal{O}_X$ ). Let  $\mathcal{E}$  be a locally projective  $\mathcal{V}_X^\infty$ -module of finite rank. Consider its stalk  $\mathcal{E}_{x_0}$  as  $\mathcal{V}_{x_0}$ -module. Then there exists a  $\mathcal{V}_{x_0}$ -module  $\mathcal{F}$  such that

$$\mathcal{E}_{x_0} \oplus \mathcal{F} \simeq \mathcal{V}_{x_0}^N$$

for some natural number  $N$ . Consider the idempotent endomorphism of the  $\mathcal{V}_{x_0}$ -module

$$e: \mathcal{V}_{x_0}^N \rightarrow \mathcal{V}_{x_0}^N$$

given by the projection onto  $\mathcal{E}_{x_0}$ . Thus  $e^2 = e$ . Notice that  $\mathcal{V}_{x_0}$  is a local ring with the maximal ideal

$$m := \{\phi \in \mathcal{V}_{x_0} \mid \phi(\{x_0\}) = 0\}.$$

Clearly  $\mathcal{V}_{x_0}/m = \mathbb{C}$ .

We have

$$\mathbb{C}^N = \mathcal{V}_{x_0}^N \otimes_{\mathcal{V}_{x_0}} (\mathcal{V}_{x_0}/m) = (\mathcal{E}_{x_0} \otimes_{\mathcal{V}_{x_0}} \mathcal{V}_{x_0}/m) \oplus (\mathcal{F} \otimes_{\mathcal{V}_{x_0}} \mathcal{V}_{x_0}/m). \quad (4)$$

Let us choose a basis

$$\xi'_1, \dots, \xi'_k, f'_1, \dots, f'_{N-k}$$

of  $\mathbb{C}^N$  such that the  $\xi'_i$ 's form a basis of the first summand in the right hand side of (4), and the  $f'_j$ 's form a basis of the second summand. Let  $\tilde{\xi}_i \in \mathcal{V}_{x_0}^N$ ,  $\tilde{f}_j \in \mathcal{V}_{x_0}^N$  be their lifts. Define finally

$$\begin{aligned} \xi_i &:= e(\tilde{\xi}_i) \in \mathcal{E}_{x_0}, \\ f_j &:= (1 - e)(\tilde{f}_j) \in \mathcal{F}. \end{aligned}$$

It is clear that

$$\begin{aligned} \xi_i &\equiv \tilde{\xi}_i \pmod{m}, \\ f_j &\equiv \tilde{f}_j \pmod{m}. \end{aligned}$$

Consider the morphism of  $\mathcal{V}_{x_0}$ -modules  $\theta: \mathcal{V}_{x_0}^k \rightarrow \mathcal{E}_{x_0}$  given by

$$\theta(\phi_1, \dots, \phi_k) := \sum_{i=1}^k \phi_i \xi_i.$$

Consider also another morphism  $\tau: \mathcal{V}_{x_0}^{N-k} \rightarrow \mathcal{F}$  given by

$$\tau(\psi_1, \dots, \psi_{N-k}) = \sum_{j=1}^{N-k} \psi_j f_j.$$

Now define a morphism of  $\mathcal{V}_{x_0}$ -modules by

$$\sigma := \theta \oplus \tau: \mathcal{V}_{x_0}^N = \mathcal{V}_{x_0}^k \oplus \mathcal{V}_{x_0}^{N-k} \rightarrow \mathcal{E}_{x_0} \oplus \mathcal{F} \simeq \mathcal{V}_{x_0}^N.$$

We claim that  $\sigma$  is an isomorphism. It is equivalent to the property that  $\det(\sigma) \in \mathcal{V}_{x_0}$  is invertible. In order to see this it suffices to show that  $(\det \sigma)(\{x_0\}) \neq 0$ . But the last condition is satisfied since

$$\sigma \otimes Id_{\mathcal{V}_{x_0}/m}: \mathcal{V}_{x_0}^N \otimes_{\mathcal{V}_{x_0}} \mathcal{V}_{x_0}/m \rightarrow \mathcal{V}_{x_0}^N \otimes_{\mathcal{V}_{x_0}} \mathcal{V}_{x_0}/m$$

is an isomorphism  $\mathbb{C}^N \rightarrow \mathbb{C}^N$  since by construction

$$\xi'_1 = \xi_1(\{x_0\}), \dots, \xi'_k = \xi_k(\{x_0\}), f'_1 = f_1(\{x_0\}), \dots, f'_{N-k} = f_{N-k}(\{x_0\})$$

form a basis of  $\mathbb{C}^N$ .

Since  $\sigma: \mathcal{V}_{x_0}^N \rightarrow \mathcal{V}_{x_0}^N$  is an isomorphism, it follows that there exists an open neighborhood  $U$  of  $x_0$  and an isomorphism of  $\mathcal{V}_U^\infty$ -modules

$$\tilde{\sigma}: (\mathcal{V}_U^\infty)^N \rightarrow (\mathcal{V}_U^\infty)^N$$

which extends  $\sigma$ , i.e.  $\sigma$  is the stalk of  $\tilde{\sigma}$  at  $x_0$ . It follows that  $\theta$  extends to an isomorphism

$$\tilde{\theta}: (\mathcal{V}_U^\infty)^k \xrightarrow{\sim} \mathcal{E}|_U$$

of  $\mathcal{V}_U^\infty$ -modules (and similarly for  $\tau$ ). Q.E.D.

**2.2 Lemma.** *Let  $\mathcal{E}$  be a locally free  $\mathcal{V}_X^\infty$ -module of finite rank  $N$ . Let  $\xi_1, \dots, \xi_k \in H^0(X, \mathcal{E})$  be chosen such that for every point  $x_0 \in X$  their images  $\bar{\xi}_1, \dots, \bar{\xi}_k$  in  $\mathcal{E} \otimes_{\mathcal{V}_X^\infty} \mathcal{V}_X^\infty / m_{x_0} \simeq \mathbb{C}^N$  form a linearly independent sequence. Consider the morphism of  $\mathcal{V}_X^\infty$ -modules*

$$f: (\mathcal{V}_X^\infty)^k \rightarrow \mathcal{E},$$

given by  $f(\phi_1, \dots, \phi_k) = \sum_{i=1}^k \phi_i \xi_i$ .

Then  $f: (\mathcal{V}_X^\infty)^k \rightarrow \text{Im}(f)$  is an isomorphism, and  $\mathcal{E}/\text{Im}(f)$  is a locally free  $\mathcal{V}_X^\infty$ -module of rank  $N - k$ .

**Proof.** Let us denote for brevity  $\mathcal{V} := \mathcal{V}_X^\infty$ . The statement is local on  $X$ . Fix  $x_0 \in X$ . We can choose  $\eta_1, \dots, \eta_{N-k} \in H^0(\mathcal{E})$  such that their images

$$\bar{\xi}_1, \dots, \bar{\xi}_k, \bar{\eta}_1, \dots, \bar{\eta}_{N-k} \in \mathcal{E} \otimes_{\mathcal{V}} \mathcal{V} / m_{x_0}$$

form a basis. Consider a morphism of  $\mathcal{V}$ -modules  $g: \mathcal{V}^k \oplus \mathcal{V}^{N-k} \rightarrow \mathcal{E}$  given by

$$g(\phi_1, \dots, \phi_k; \psi_1, \dots, \psi_{N-k}) = \sum_{i=1}^k \phi_i \xi_i + \sum_{j=1}^{N-k} \psi_j \eta_j.$$

Clearly  $g|_{\mathcal{V}^k} \equiv f$ . In a neighborhood of  $x_0$  we may and will identify  $\mathcal{E} \simeq \mathcal{V}^N$ . Then

$$g: \mathcal{V}^k \oplus \mathcal{V}^{N-k} = \mathcal{V}^N \rightarrow \mathcal{V}^N.$$

It is easy to see that the map

$$g \otimes \text{Id}_{\mathcal{V}/m_{x_0}}: \mathbb{C}^N \simeq (\mathcal{V}/m_{x_0})^N \rightarrow \mathbb{C}^N \simeq (\mathcal{V}/m_{x_0})^N$$

is an isomorphism. Hence  $(\det g)(\{x_0\}) \neq 0$ . It follows that  $\det g \in \mathcal{V}$  is invertible in a neighborhood of  $x_0$ . Hence  $g$  is an isomorphism in a neighborhood of  $x_0$ . This implies the lemma immediately. Q.E.D.

**2.3 Lemma.** *Let  $P$  be a locally free  $\mathcal{V}_X^\infty$ -module of finite rank. Then for any  $\mathcal{V}_X^\infty$ -module  $A$ ,*

$$\text{Ext}_{\mathcal{V}_X^\infty\text{-mod}}^i(P, A) = 0 \text{ for } i > 0.$$

**Proof.** We abbreviate again  $\mathcal{V} := \mathcal{V}_X^\infty$ . First notice that in the category of  $\mathcal{V}$ -modules the following two functors

$$F, G: \mathcal{V}\text{-mod} \rightarrow \text{Vect}$$

are naturally isomorphic:

$$\begin{aligned} F(A) &= \text{Hom}_{\mathcal{V}\text{-mod}}(P, A), \\ G(A) &= H^0(X, P^* \otimes_{\mathcal{V}} A), \end{aligned}$$

where  $P^* := \underline{\text{Hom}}_{\mathcal{V}\text{-mod}}(P, \mathcal{V})$  is the inner Hom as usual. Indeed the natural morphism

$$P^* \otimes_{\mathcal{V}} A = \underline{\text{Hom}}_{\mathcal{V}\text{-mod}}(P, \mathcal{V}) \otimes_{\mathcal{V}} A \rightarrow \underline{\text{Hom}}_{\mathcal{V}\text{-mod}}(P, A)$$

is an isomorphism of sheaves. Taking global sections, we get an isomorphism

$$H^0(X, P^* \otimes_{\mathcal{V}} A) \xrightarrow{\sim} H^0(X, \underline{\text{Hom}}_{\mathcal{V}\text{-mod}}(P, A)).$$

But the last space is equal to  $\text{Hom}_{\mathcal{V}\text{-mod}}(P, A)$  (see [10], Ch. II, §1, Exc. 1.15).

Consequently  $F$  and  $G$  have isomorphic derived functors. Hence

$$\text{Ext}_{\mathcal{V}\text{-mod}}^i(P, A) \simeq H^i(X, P^* \otimes_{\mathcal{V}} A).$$

But the last group vanishes for  $i > 0$  by [3], Lemma 5.1.2. Q.E.D.

**2.4 Corollary.** *Let*

$$0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$$

*be a short exact sequence of  $\mathcal{V}_X^\infty$ -modules. If  $C$  is locally free of finite rank, then this exact sequence splits.*

**Proof.** Indeed  $\text{Ext}_{\mathcal{V}\text{-mod}}^1(C, A) = 0$  by Lemma 2.3. Q.E.D.

**2.5 Proposition.** *Let  $X$  be a compact manifold. Let  $\mathcal{E}$  be a locally free  $\mathcal{V}_X^\infty$ -module of finite rank. Then there exists another locally free  $\mathcal{V}_X^\infty$ -module  $\mathcal{H}$  of finite rank such that*

$$\mathcal{E} \oplus \mathcal{H} \simeq (\mathcal{V}_X^\infty)^N.$$

**Proof.** Let us choose a finite open covering  $X = \cup_\alpha U_\alpha$  such that the sheaf  $\mathcal{E}|_{U_\alpha}$  is free for each  $\alpha$ . Let  $\{\phi_\alpha\}$  be a partition of unity in the algebra of valuations subordinate to this covering (it exist by [3], Proposition 6.2.1). We can find a finite dimensional subspace  $L_\alpha \subset H^0(U_\alpha, \mathcal{E})$  which generates  $\mathcal{E}|_{U_\alpha}$  as  $\mathcal{V}_{U_\alpha}^\infty$ -module. Consider  $\phi_\alpha \cdot L_\alpha \subset H^0(X, \mathcal{E})$  (where all sections are extended by zero outside of  $U_\alpha$ ). Then the finite dimensional subspace

$$L := \sum_{\alpha} \phi_\alpha \cdot L_\alpha \subset H^0(X, \mathcal{E})$$

generates  $\mathcal{E}$  as  $\mathcal{V}$ -module (indeed at every  $x \in X$  there exists an  $\alpha$  such that  $\phi_\alpha$  is invertible in a neighborhood of  $x$ ).

Let us choose a basis  $\xi_1, \dots, \xi_s$  of  $L$ . Consider the morphism of  $\mathcal{V}$ -modules  $F: \mathcal{V}^s \rightarrow \mathcal{E}$  given by

$$F(\phi_1, \dots, \phi_s) = \sum_{i=1}^s \phi_i \xi_i.$$

Clearly  $F$  is an epimorphism of  $\mathcal{V}$ -modules. Let  $\mathcal{A} := \text{Ker}(F)$ . By Corollary 2.4 the short exact sequence

$$0 \rightarrow \mathcal{A} \rightarrow \mathcal{V}^s \rightarrow \mathcal{E} \rightarrow 0$$

splits. Thus  $\mathcal{E} \oplus \mathcal{A} \simeq \mathcal{V}^s$ . Hence  $\mathcal{A}$  is locally projective. Hence  $\mathcal{A}$  is locally free by Proposition 2.1. Q.E.D.

Let us denote by  $\text{Proj}_f \mathcal{V}_X^\infty - \text{mod}$  (or just  $\text{Proj}_f \mathcal{V} - \text{mod}$ ) the full subcategory of  $\mathcal{V} - \text{mod}$  consisting of locally free  $\mathcal{V}$ -modules of finite rank. Let us denote by  $\text{Proj}_f V^\infty(X) - \text{mod}$  the category of projective  $V^\infty(X)$ -modules of finite rank.

**2.6 Theorem.** *Let  $X$  be a compact manifold. Then the functor of global sections*

$$\Gamma: \text{Proj}_f \mathcal{V}_X^\infty - \text{mod} \rightarrow \text{Proj}_f V^\infty(X) - \text{mod}$$

*is an equivalence of categories.*

**Proof.** We denote again by  $\mathcal{V} := \mathcal{V}_X^\infty$ . Let  $\mathcal{A}, \mathcal{B} \in \text{Proj}_f \mathcal{V} - \text{mod}$ . First let us show that

$$\text{Hom}_{\mathcal{V} - \text{mod}}(\mathcal{A}, \mathcal{B}) = \text{Hom}_{V^\infty(X) - \text{mod}}(\Gamma(\mathcal{A}), \Gamma(\mathcal{B})).$$

Both  $\text{Hom}$  functors respect finite direct sums with respect to both arguments. Since  $\mathcal{A}, \mathcal{B}$  are direct summands of free  $\mathcal{V}$ -modules by Proposition 2.5 we may assume that  $\mathcal{A} = \mathcal{B} = \mathcal{V}$ . But clearly

$$\begin{aligned} \text{Hom}_{\mathcal{V} - \text{mod}}(\mathcal{V}, \mathcal{V}) &= V^\infty(X), \\ \text{Hom}_{V^\infty(X)}(V^\infty(X), V^\infty(X)) &= V^\infty(X). \end{aligned}$$

Thus  $\Gamma$  is fully faithful.

Let us define a functor in the opposite direction (the localization functor),

$$G: \text{Proj}_f V^\infty(X) - \text{mod} \rightarrow \text{Proj}_f \mathcal{V} - \text{mod},$$

by  $G(A) := A \otimes_{V^\infty(X)} \mathcal{V}$ .  $G$  is also fully faithful: it commutes with direct sums, and for trivial  $V^\infty(X)$ -modules the statement is obvious.

The functors  $F \circ G$  and  $G \circ F$  are naturally isomorphic to the identity functors. Q.E.D.

### 3 Isomorphism classes of bundles over valuations.

Recall that the sheaf of smooth valuations  $\mathcal{V}_X^\infty$ , which we will denote for brevity by  $\mathcal{V}$ , has a canonical filtration by subsheaves,

$$\mathcal{V} = \mathcal{W}_0 \supset \mathcal{W}_1 \supset \cdots \supset \mathcal{W}_n.$$

This filtration is compatible with the product, and  $\mathcal{V}/\mathcal{W}_1 \simeq \mathcal{O}_X$  canonically [3]. Let us fix a natural number  $N$ . Let us denote by  $\underline{GL}_N(\mathcal{V})$  (resp.  $\underline{GL}_N(\mathcal{O}_X)$ ) the sheaf on  $X$  of invertible

$N \times N$  matrices with entries in  $\mathcal{V}$  (resp.  $\mathcal{O}_X$ ). We have a natural homomorphism of sheaves of groups

$$\underline{GL_N(\mathcal{V})} \rightarrow \underline{GL_N(\mathcal{O}_X)}. \quad (5)$$

It is well known that isomorphism classes of usual vector bundles are in bijective correspondence with the (Cech) cohomology set  $H^1(X, \underline{GL_N(\mathcal{O}_X)})$ . Similarly it is clear that locally free  $\mathcal{V}$ -modules of rank  $N$  are in bijective correspondence with the set  $H^1(X, \underline{GL_N(\mathcal{V})})$ . The main result of this section is

**3.1 Theorem.** *Let  $X$  be a compact manifold. The natural map*

$$H^1(X, \underline{GL_N(\mathcal{V})}) \rightarrow H^1(X, \underline{GL_N(\mathcal{O}_X)})$$

*induced by (5) is a bijection. Thus, if  $X$  is compact, the isomorphism classes of rank  $N$  locally free  $\mathcal{V}$ -modules are in natural bijective correspondence with isomorphism classes of rank  $N$  vector bundles.*

We will need some preparations before the proof of the theorem. First we observe that  $\underline{GL_N(\mathcal{V})}$  has a natural filtration by subsheaves of normal subgroups

$$\underline{GL_N(\mathcal{V})} =: \mathcal{K}_0 \supset \mathcal{K}_1 \supset \cdots \supset \mathcal{K}_n,$$

where for any  $i > 0$ ,  $\mathcal{K}_i(U) := \{\xi \in \underline{GL_N(\mathcal{V})}(U) \mid \xi \equiv I \bmod \mathcal{W}_i(U)\}$  for any open subset  $U \subset X$ . We have the canonical isomorphisms of sheaves of groups:

$$\mathcal{K}_0/\mathcal{K}_1 \simeq \underline{GL_N(\mathcal{O}_X)}, \quad (6)$$

$$\mathcal{K}_i/\mathcal{K}_{i+1} \simeq (\mathcal{W}_i/\mathcal{W}_{i+1})^N \text{ for } i > 0. \quad (7)$$

We have to remind some general results due to Grothendieck [9]. Let  $\underline{G}$  be a sheaf of groups (not necessarily abelian) on a topological space  $X$ . Let  $\underline{F} \subset \underline{G}$  be a subsheaf of *normal* subgroups. Let  $\underline{H} := \underline{G}/\underline{F}$  be the quotient sheaf, which is also a sheaf of groups. Notice that the sheaf of groups  $\underline{G}$  acts on  $\underline{F}$  by conjugations.

Let  $E'$  be a  $\underline{G}$ -torsor. Let  $c' := [E'] \in H^1(X, \underline{G})$  be its class. Define a new sheaf  $\underline{F}(E')$  to be the sheaf associated to the presheaf

$$U \mapsto \underline{F}(U) \times_{\underline{G}(U)} E'(U).$$

Since  $\underline{G}$  acts on  $\underline{F}$  by automorphisms, it follows that  $\underline{F}(E')$  is a sheaf of groups.

Grothendieck [9] has constructed a map

$$i_1: H^1(X, \underline{F}(E')) \rightarrow H^1(X, \underline{G}),$$

and he has shown (see Corollary after Proposition 5.6.2 in [9]) that the set of classes  $c \in H^1(X, \underline{G})$  which have the same image as  $c'$  under the natural map

$$H^1(X, \underline{G}) \rightarrow H^1(X, \underline{H})$$

is equal to the image of the map  $i_1$ . In particular we deduce immediately the following claim.



**3.2 Claim.** *If, for any  $\underline{G}$ -torsor  $E'$ ,*

$$H^1(X, \underline{F}(E')) = 0$$

*then the natural map  $H^1(X, \underline{G}) \rightarrow H^1(X, \underline{H})$  is injective.*

We will need the following proposition.

**3.3 Proposition.** *For any  $1 \leq i < j$  and for any  $\mathcal{K}_0/\mathcal{K}_j$ -torsor  $E'$ , one has*

$$H^1(X, (\mathcal{K}_i/\mathcal{K}_j)(E')) = 0.$$

**Proof.** The proof is by induction in  $j - i$ . Assume first that  $j - i = 1$ .  $(\mathcal{K}_i/\mathcal{K}_{i+1})(E')$  is a sheaf of  $\mathcal{O}_X$ -modules; this follows easily from (7) and the fact that  $\mathcal{W}_i/\mathcal{W}_{i+1}$  is a sheaf of  $\mathcal{O}_X$ -modules. Hence  $(\mathcal{K}_i/\mathcal{K}_{i+1})(E')$  is acyclic.

Assume now that  $j - i > 1$ . Then we have a short exact sequence of sheaves

$$1 \rightarrow (\mathcal{K}_j/\mathcal{K}_{j-1})(E') \rightarrow (\mathcal{K}_i/\mathcal{K}_j)(E') \rightarrow (\mathcal{K}_i/\mathcal{K}_{j-1})(E') \rightarrow 1.$$

Hence we have an exact sequence of pointed sets (see [9], Section 5.3),

$$H^1(X, (\mathcal{K}_j/\mathcal{K}_{j-1})(E')) \rightarrow H^1(X, (\mathcal{K}_i/\mathcal{K}_j)(E')) \rightarrow H^1(X, (\mathcal{K}_i/\mathcal{K}_{j-1})(E')).$$

The first and the third terms of the last sequence vanish by the induction assumption. Hence the middle term vanishes too. Proposition is proved. Q.E.D.

We easily deduce a corollary.

**3.4 Corollary.** *The natural map*

$$H^1(X, \underline{GL}_N(\mathcal{V})) \rightarrow H^1(X, \underline{GL}_N(\mathcal{O}_X))$$

*is injective.*

**Proof.** By Proposition 3.3  $H^1(X, \mathcal{K}_1(E')) = 0$  for any  $\mathcal{K}_0$ -torsor  $E'$ . Hence, by Claim 3.2, the map  $H^1(X, \mathcal{K}_0) \rightarrow H^1(X, \mathcal{K}_0/\mathcal{K}_1)$  is injective. Q.E.D.

We will need a few more results from [9]. Assume  $X$  is a paracompact topological space (remind that all our manifolds are always assumed to be paracompact). Let  $\underline{G}$  be a sheaf of groups on  $X$  as before. Let  $\underline{F} \triangleleft \underline{G}$  be a subsheaf of *normal abelian* subgroups. Let  $\underline{H} := \underline{G}/\underline{F}$  be the quotient sheaf as before. The action of  $\underline{G}$  on  $\underline{F}$  by conjugation induces in this case an action of  $\underline{H}$  on  $\underline{F}$ . For any  $\underline{H}$ -torsor  $E''$  one has the sheaf  $\underline{F}(E'')$  defined similarly as before. This is a sheaf of abelian groups since  $\underline{F}$  is, and  $\underline{H}$  acts on  $\underline{F}$  by automorphisms. Grothendieck ([9], Section 5.7) has constructed an element

$$\delta E'' \in H^2(X, \underline{F}(E''))$$

with the following property:  $\delta E''$  vanishes if and only if the class  $[E''] \in H^1(X, \underline{H})$  lies in the image of the canonical map  $H^1(X, \underline{G}) \rightarrow H^1(X, \underline{H})$ .

In order to apply this result in our situation we will need two lemmas.

**3.5 Lemma.** *For any  $i > 0$  and any  $\mathcal{K}_0/\mathcal{K}_i$ -torsor  $E''$ ,*

$$H^2(X, (\mathcal{K}_i/\mathcal{K}_{i+1})(E'')) = 0.$$

**Proof.** It is easy to see that  $(\mathcal{K}_i/\mathcal{K}_{i+1})(E'')$  is a sheaf of  $\mathcal{O}_X$ -modules. Hence it is acyclic. Q.E.D.

**3.6 Lemma.** *For any  $i \geq 0$  the natural map*

$$H^1(X, \mathcal{K}_0/\mathcal{K}_{i+1}) \rightarrow H^1(X, \mathcal{K}_0/\mathcal{K}_i)$$

*is onto.*

**Proof.** Let  $c'' \in H^1(X, \mathcal{K}_0/\mathcal{K}_i)$  be an arbitrary element. Let  $E''$  be a  $\mathcal{K}_0/\mathcal{K}_i$ -torsor representing  $c''$ . Consider the element  $\delta E'' \in H^2(X, (\mathcal{K}_i/\mathcal{K}_{i+1})(E''))$ . Since the last group vanishes by Lemma 3.5, by the above mentioned result of Grothendieck,  $c''$  lies in the image of  $H^1(X, \mathcal{K}_0/\mathcal{K}_{i+1})$ . Lemma is proved. Q.E.D.

**3.7 Corollary.** *The natural map*

$$H^1(X, \underline{GL_N(\mathcal{V})}) \rightarrow H^1(X, \underline{GL_N(\mathcal{O}_X)})$$

*is onto.*

**Proof.** The map in the statement factorizes into the sequence of maps

$$\begin{aligned} H^1(X, \underline{GL_N(\mathcal{V})}) &= H^1(X, \mathcal{K}_0) \rightarrow H^1(X, \mathcal{K}_0/\mathcal{K}_n) \rightarrow H^1(X, \mathcal{K}_0/\mathcal{K}_{n-1}) \rightarrow \dots \\ &\dots \rightarrow H^1(X, \mathcal{K}_0/\mathcal{K}_1) = H^1(X, \underline{GL_N(\mathcal{O}_X)}) \end{aligned}$$

where all the maps are surjective by Lemma 3.6. Hence their composition is onto too. Q.E.D.

Now Theorem 3.1 follows immediately from Corollaries 3.4 and 3.7.

**3.8 Remark.** Theorem 3.1 has the following immediate consequence. For a compact manifold  $X$  we can construct a  $K$ -ring generated by finitely generated projective  $V^\infty(X)$ -modules in the standard way. Namely as a group it is equal to the quotient of the free abelian group generated by isomorphism classes of such modules by the relations

$$[M \oplus N] = [M] + [N].$$

The product is induced by the tensor product of such  $V^\infty(X)$ -modules. Then Theorem 3.1 implies that there is a canonical isomorphism of this  $K$ -ring with the classical topological  $K^0$ -ring (see [6]) constructed from vector bundles.

**3.9 Remark.** The main results of this paper are of general nature. It would be interesting to have concrete geometric examples of  $\mathcal{V}_X^\infty$ -modules; in the classical case of  $\mathcal{O}_X$ -modules we have the tangent bundle and its tensor powers.

As a first small step in this direction let us mention the following construction. Let  $\mathcal{L}$  be a flat vector bundle over a manifold  $X$ . By an abuse of notation, we will also denote by  $\mathcal{L}$  the sheaf of its locally constant sections. Let  $\underline{\mathbb{C}}$  be the constant sheaf of  $\mathbb{C}$ -vector spaces. Consider the  $\mathcal{V}_X^\infty$ -module defined by

$$\tilde{\mathcal{L}} := \mathcal{L} \otimes_{\underline{\mathbb{C}}} \mathcal{V}_X^\infty$$

where we consider  $\mathcal{V}_X^\infty$  as  $\underline{\mathbb{C}}$ -module via the imbedding  $\underline{\mathbb{C}} \hookrightarrow \mathcal{V}_X^\infty$  where 1 goes to the Euler characteristic. It is easy to see that  $\tilde{\mathcal{L}}$  is a locally free  $\mathcal{V}_X^\infty$ -module.

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